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# A Discussion of Causality and the Lorentz Group

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*Abstract*

Zeeman (1964) has shown that the group of automorphisms for the relation of causality on Minkowski space is that generated by the orthochronous Poincaré (Halpern, 1968) group and dilatations. Here we prove that the group of automorphisms that preserve the time-like vectors of Minkowski space normwise is the complete Poincaré group. We prove that the timelike structure within the null cone of a single event does define the whole structure of Minkowski space. Further, it is shown that only inertial observers can use Minkowski space to describe space-time.

## *Introduction*

Let $M$ denote Minkowski space, that is $R^4$ with pseudo norm

$$\|y - x\|^2 = \varepsilon\{(y_0 - x_0)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2 - (y_3 - x_3)^2\}$$

and Euclidean topology. An event $y$ is said to be in the future of an event $x$ if the vector $y - x$ is timelike; that is, if $\varepsilon\|y - x\|^2 > 0$, with $x_0 < y_0$. This relation is written $x < y$. If $f$ is an injective mapping of $M$ into itself it is said to be a causal automorphism if $f$ and $f^{-1}$ preserve this relation of causality

$$x < y \Leftrightarrow fx < fy \qquad \text{for all such } x, y \in M$$

The causal automorphisms form a group called the causality group. Zeeman has shown that the causality group is generated by the orthochronous Poincaré group and dilatations.

Here we study the group of automorphisms $G$ of $M$ that preserve the norms of timelike vectors, making no assumptions about linearity or continuity. $f$ is an element of this group if it is injective and such that

$$\varepsilon\|y - x\|^2 = \varepsilon'\|fy - fx\|^2 \qquad \text{for } x, y \in M$$

such that $\varepsilon\|y - x\|^2 > 0$. The complete Poincaré group will of course be a subgroup of $G$. This investigation shows that $G$ is in fact the complete Poincaré group. Hence preserving the norms of timelike vectors automatically preserves all norms.

Our investigation differs from Zeeman's in that we do demand preservation of timelike norms but relax his demand of global causality. The techniques used here involve the 'inverse triangle inequality' property of the Minkowski metric, and it may be possible to generalize the results to some curved space-times. The Minkowski metric in effect defines two such inequalities, one on triangles composed of timelike lines and one made up of spacelike lines.

*Lemma* 1. For $x, y, z \in M$, $\varepsilon\|y-x\|^2 > 0$, $\varepsilon'\|z-x\|^2 > 0$, then $z \in xy$ (that is, $z$ lies on the line $xy$ between $x$ and $y$) if and only if $\|y-x\| = \|z-x\| + \|y-z\|$.

*Proof.* This result is the basis of the clock paradox discussion in special relativity. If $z$ lies on $xy$, in the future of one event and in the past of the other, then we know that $\|y-x\| = \|z-x\| + \|y-z\|$. Otherwise $\|y-x\| > \|z-x\| + \|y-z\|$. This latter case accounts for the difference in the ages of twins starting at $x$ on being reunited at $y$, one having taken the inertial path $xy$ and the other the non-inertial path $xzy$.

*Theorem* 1. $f$ transforms timelike lines into timelike lines.

*Proof.* Let $x, y \in M$, $\varepsilon\|y-x\|^2 > 0$ and $z \in xy$.

$$\|y-x\| = \|z-x\| + \|y-z\| = \|fz-fx\| + \|fy-fz\|$$

But $\|y-x\| = \|fy-fx\|$. Therefore,

$$\|fy-fx\| = \|fz-fx\| + \|fy-fz\|$$

By the previous lemma, $fz \in fx\,fy$, thus $f(xy)$ is a straight line. Its timelike nature follows from the definition of $f$.

*Theorem* 2. $f$ is a homeomorphism of $M$.

*Proof.* $f$ is injective. Let

$$U = \{y : 0 < \varepsilon\|y-x\|^2 < r,\ x, y \in M,\ r \in R\}$$

be a subbasic open neighbourhood for a topology on $M$. The topology thus generated is the Euclidean topology. $f$ and $f^{-1}$ preserve such subbasic open sets,

$$fU = \{z : 0 < \varepsilon'\|z-fx\|^2 < r,\ x, z \in M,\ r \in R\}$$

Thus $f$ is a homeomorphism of $M$.

*Lemma* 2. $f$ preserves or reverses the causality relation on a timelike line.

*Proof.* Let $h$ be the homeomorphism from a given timelike line to the time axis defined by $hz = z_0$, for $z$ an arbitrary element of the timelike line.

Then $h$ defines causality on the timelike line, $x < y$ on the line if and only if $hx < hy$. Thus $h$ defines a correspondence between the causality relation on the line and the inequality relation on the time axis. $hfh^{-1}$ is a homeomorphism of the time axis onto itself. It will either preserve or reverse the inequality relation on the time axis. The former case corresponds to $f$ preserving causality on the given time line, the latter to $f$ reversing causality.

*Theorem* 3. $f$ either preserves or reverses causality on all timelike lines.

*Proof.* We firstly prove this result for two timelike lines passing through a common event $x$. Let $Y$ and $Z$ be two such lines and let $y$ and $z$ be events on these lines such that $x < y$, $x < z$, $y < z$. Suppose $f$ preserves causality on $Z$ but reverses it on $Y$, so that $fy < fx$, $fx < fz$. It follows from the transitivity of the causality relation that $fy < fz$. Thus we have

$$\|y - x\| + \|z - y\| < \|z - x\|$$

$$\|fz - fx\| + \|fx - fy\| < \|fz - fy\|$$

this latter inequality giving

$$\|z - x\| + \|x - y\| < \|z - y\|$$

leading to a contradiction. Hence $f$ either preserves causality on all timelike lines passing through a single event, or reverses it.

Consider two non-intersecting timelike lines. There exists a timelike line that intersects both. By transitivity on each pair of intersecting lines it can be seen that $f$ will either preserve causality on both the non-intersecting lines or reverse it. At this point we know from Zeeman's work that $G$ is the complete Poincaré group, since this is the subset of the causality preserving or reversing group that preserves timelike norms. However, we derive this result using techniques which differ from Zeeman's.

*Theorem* 4. $f$ preserves null cones. That is, if $y$ is an element of the null cone at $x$, $fy$ is an element of the null cone at $fx$.

*Proof.* In the Euclidean topology the null cone at $x$, together with $x$, is the boundary of the set making up the future and past of $x$. Since $f$ is a homeomorphism that preserves this set, it must also preserve its boundary.

*Theorem* 5. $f$ transforms null lines into null lines.

*Proof.* Let $z \in xy$, a null line. By the previous theorem,

$$\|fy - fx\| = \|fz - fx\| = \|fz - fy\| = 0$$

Thus $fy$ and $fz$ are elements of the null cone at $fx$ and $fz$ is an element of the null cone at $fy$. Since the intersection of the null cones at $fx$ and $fy$ is the line $fx\,fy$, $fz$ must lie on $fx\,fy$.

*Lemma* 3. Let $Y$ and $Z$ be timelike lines through a common event $x$. If $P$ is a timelike line through $x$, lying in the plane of $Y$ and $Z$, then $fP$ lies in the plane of $fY$ and $fZ$.

*Proof.* Let $n$ be an arbitrary element of $fP$. Then $f^{-1}n$ is an element of $P$. Hence there exists a timelike line $Q$ through $f^{-1}n$ cutting $Y$ at $m$ and $Z$ at $r$. $n$ lies on $fQ$, which passes through $fm$ and $fr$, elements of $fY$ and $fZ$ respectively. Hence $n$ lies in the plane of $fY$ and $fZ$. Thus $fP$ lies in this plane.

*Theorem* 6. $f$ transforms spacelike lines into spacelike lines.

*Proof.* Let $\varepsilon\|y-x\|^2<0$, $z\in xy$ and $p$ be an event such that $p<x$ and $p<y$. Then $z$ lies in the plane $pxy$ and thus, by the previous lemma, $fz\in$ plane $fp\,fx\,fy$. Let $p^1$ be a second such event, not an element of the plane $pxy$. Then $fz\in$ plane $fp^1\,fx\,fy$ also. Thus $fz$ lies in the intersection of these two distinct planes, which is the line $fx\,fy$. Thus $f$ maps spacelike lines into lines. $fx\,fy$ must be spacelike since it cannot be timelike or null.

*Theorem* 7. $G$ is the complete Poincaré group.

*Proof.* The elements of $G$ are injective and projective (preserve straight lines), thus $G$ must be a group generated by a subgroup of the full linear group on $M$ and the group of uniform displacements of the origin. The subgroup of the full linear group that preserves timelike vectors and null vectors is the conformal linear group. If $f$ is a conformal linear mapping then

$$\varepsilon\|y-x\|^2=\gamma\varepsilon'\|fy-fx\|^2$$

$\gamma$ a scalar, for every $x, y\in M$. But for $y-x$ timelike $\gamma=1$. Thus $f$ must be a linear isometry. Thus $G$ is the group generated by the group of linear isometries of $M$ and the group of uniform displacements. It is thus the complete Poincaré group.

We now go on to prove that defining the mapping $f$ inside the null cone at one event defines it uniquely on $M$.

*Theorem* 8. Let $\mathscr{F}$ be the interior of the null cone at $x'\in M$ and let $f\colon\mathscr{F}\to M$ such that

$$\varepsilon\|y-x\|^2=\varepsilon'\|fy-fx\|^2 \qquad \text{for all } y, x\in\mathscr{F}$$

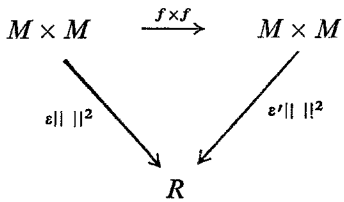
such that $\varepsilon\|y-x\|^2>0$. Then $f$ is the restriction to $\mathscr{F}$ of a unique element of the complete Poincaré group.

*Proof.* It can be shown, as previously, that $f$ transforms line segments of $M$ that lie within $\mathscr{F}$ into line segments. Thus $f$ preserves the straight line structure within $\mathscr{F}$. Let $xy$ be a line in $\mathscr{F}$. It defines a unique line $c$ in $M$.

$f(xy)$ defines a unique line $c^1$ in $M$. Define $F: M \rightarrow M$ such that $Fc = c^1$. This defines $F$ uniquely on $M$, since every element of $M$ lies on two distinct lines that intersect $\mathscr{F}$. The restriction of $F$ to $\mathscr{F}$ is $f$. Suppose $m$ is a line in $M$ that does not intersect $\mathscr{F}$. It is always possible to find two planes that intersect in $m$, these planes both intersecting $\mathscr{F}$. $F$ preserves these planes hence $F$ preserves the linearity of $m$. Thus $F$ is projective. It must thus be an element of the group generated by the full linear group and the group of uniform displacements. It is thus continuous. $F$ must thus map the null cone at an arbitrary point $y$ into the null cone at $Fy$. Therefore $F$ is conformal. Since $F$ preserves the norms of timelike vectors at $x'$ the conformal factor must be unity, proving that $F$ is an isometry and thus an element of the complete Poincaré group.

## *General Discussion*

$f$ is an element of the complete Poincaré group if and only if the following diagram is commutative.




Here we have seen that defining $f$ in such a manner that it is commutative on the subset

$$\{(x, y): x, y \in M, x \text{ fixed}, y \text{ such that } \varepsilon\|y - x\|^2 > 0\}$$

of $M \times M$ makes it commutative everywhere.

This result leads one to speculate about similar results for a positive definite metric on $R^n$. If $f$ is an isometry on $R^n$ then it is an element of the group generated by the full linear group and group of uniform displacements. However, if $f$ is an injective mapping on $R^n$ which preserves the metric only on certain subsets, it need not be an isometry. What interesting classes of subsets need $f$ preserve the metric on for it to be an isometry?

We now prove that if we define $f$ to be a homeomorphism on $M$ and norm preserving on null cones it is in fact an element of the group generated by the complete Poincaré group and dilatations. We show that $f$ preserves or reverses causality on all null lines. Firstly, since $f$ preserves null cones it transforms null lines into null lines (Theorem 5). Further, it preserves or reverses causality on a single null line, since it is a homeomorphism. We now show that it preserves or reverses causality on null lines that intersect. Let $xy$ and $xz$ be such lines with $x <\cdot\, y$ and $x <\cdot\, z$; that is, $x$ causally preceding both $y$ and $z$. (We use $<\cdot$ for lightlike causality.) Suppose $f$ does not preserve causality; let $fx <\cdot\, fy$ but $fz <\cdot\, fx$. The null cones at $y$ and $z$ intersect in disjoint sets $U$ and $V$, the elements of $U$ being in the future null cones of $y$ and $z$ while those of $V$ are in their past null cones; $x \in V$. However, the null cones of $fz$ and $fy$ intersect in the connected set $f(U \cup V)$ which is on the past null cone of $y$ but the future null cone of $z$. Since $f$ is a homeomorphism it preserves the topological property of connectedness, leading to a contradiction. Thus $f$ must preserve or reverse causality on both $xy$ and $xz$; causality is preserved or reversed on null lines that intersect. By transitivity we again see that $f$ must preserve or reverse causality on all null lines. Thus, from Zeeman's work we see that $f$ is an element of the group generated by the complete Poincaré group and dilatations. Conversely, every element of this group is a homeomorphism and does preserve null norms. It is conjectured that the conditions may be weakened by replacing the homeomorphism requirement by just an injective requirement.

Physically, of course, preserving null norms corresponds to preserving the velocity of light, and this is a discussion of transformations on Minkowski space that preserve the magnitude of the velocity of light. To derive the Poincaré group in special relativity one usually uses the principle of relativity (all laws of nature are identical in all inertial systems of reference) to show that invariance of null infinitesimals leads to the invariance of magnitude of all infinitesimals (see, for example, Landau & Lifshitz, 1951). This invariance soon leads to the Poincaré group (see, for example, Synge, 1956). (Assuming that all observers use the same scale makes the conformal factor unity.) Our result implies that one need not use the principle of relativity, the Poincaré group follows from the invariance of the velocity of light. The homeomorphic requirement presents no real limitation on the physical interpretation of the result, as any such $f$ would correspond to a coordinate transformation relating the Minkowski space representation of two distinct observers and any function on or into $M$ that would appear continuous to the one observer would have to do so for the other observer. Thus let $O$ and $O'$ be distinct observers, $O$ being an inertial observer, both using Minkowski coordinates. Then, if and only if $O'$ can describe the paths of photons in his Minkowski coordinate system by null lines and the two coordinate systems are homeomorphically related, is $O'$ himself an inertial observer.

We have seen that, mathematically, the group that preserves timelike norms is a proper subgroup of that group that homeomorphically preserves null norms. There is no type of triangle inequality that has to be preserved in preserving null norms, whereas there is when timelike norms are preserved. The property of the metric that is preserved for the null norms is more akin to its semimetric part (Blumenthal, 1953). (It is the triangle inequality part of the metric that limits the metric function, in that the number assigned to one pair of points is related to that assigned to other pairs.)

There is no analog of Theorem 8 for homeomorphisms that preserve null norms. Defining $f$ to be null cone preserving at one event does not define a unique element of the causality preserving or reversing group. Let $N$ be the null cone at $x$, $M$ the null cone at $y$, $f$ an injective mapping of

$N \to M$ with $fx = y$. Then $f$ can homeomorphically expand one null line and contract another. $f$ does define a subclass of the causality-preserving or reversing group, each element of which maps $N \to M$ and $x \to y$. If $L$ is an arbitrary Lorentz transformation, $E$ an arbitrary dilatation and $D$ the uniform displacement operator that maps $x \to y$, then $DEL$, the composite mapping, is an element of this class, and every element of the causality-preserving or reversing group that preserves $N$ thus is of this type. However, $f$ need not be the restriction of such a mapping to $N$.

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